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VELOCITY FIELD EQUATIONS AND STRAIN LOCALIZATION IN RIGID–PLASTIC MATERIALS

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Abstract—The velocity field equations are set up for a standard rigid–plastic material. They allow a direct determination of the velocity field, and therefore correspond to the Navier–Stokes equations of a Newtonian fluid. Under conditions of plane strain or axisymmetric deformation, the velocity field equations lead, using the Lévy–Huber–von Mises flow law, to the principal lines—i.e. to the trajectories of the principal strain-rates as characteristics. However, an incipient strain localization, combined with a velocity discontinuity, cannot be expected to occur along a principal line. Rather, a degeneration of the velocity field equations, shown to exist for certain rate-sensitive materials, might be responsible for such a localization effect.

1. INTRODUCTION

Velocity field equations are differential equations which allow a direct determination of the velocity field in a continuum. For a Newtonian fluid, the velocity field equations are the Navier–Stokes equations. They are equivalent to the Navier equations of elasticity, provided that the displacements are considered instead of the velocities. For an elastic–plastic continuum with strain sensitivity, the velocity field equations have been given by Lippmann (1986) with respect to the actual configuration of the body.

In the present paper, the velocity field equations will be derived for an incompressible† rigid–plastic material, with or without strain hardening, rate hardening or temperature sensitivity, again with respect to the actual configuration. They become, as usual, partial differential equations of the second order in the velocity terms, and of the first order in the hydrostatic stress. The characteristics must therefore be defined as lines or interfaces across which either the first order spatial derivatives of the hydrostatic stress or the second order spatial derivatives of the velocity, i.e. the first order spatial derivatives of the strain rates, may become nonunique or discontinuous, even if the hydrostatic stress itself or the first order derivatives of the velocities are continuous. If a permanently continuous field of strain rates is assumed, then the strain field and the equivalent strain are also continuous. This continuity holds also for the temperature field, for, its gradient may be discontinuous only across jump interfaces of the velocity (Becker *et al.*, 1987). Velocity jumps will not be admitted in the following sections. The temperature and the pre-strain will therefore be treated as prescribed continuous fields, although they may be initially unknown.

As an application, the characteristics will be determined for plain strain as well as for axially symmetric plastic flow, based on the Lévy–Huber–von Mises flow law. In both cases, the characteristics are the trajectories of principal strain rates—the so-called principal lines. This is in complete contrast to classical plasticity, where the plane strain characteristics are the slip lines, while no characteristics exist for axisymmetric deformation (Parsons, 1956). This has nothing to do with the observation that—on the basis of Tresca's yield condition and the associated flow rule—the principal lines, in addition to the slip lines, can in certain cases actually become the characteristics (cf. Lippmann, 1981).

The difference may formally be explained as follows.

In classical plane strain plasticity without any strain rate sensitivity, the stress field and the velocity field are uncoupled. The slip line characteristics then refer primarily to the stress field. The equations for the velocities cannot be set up prior to the determination of the stress field, and they are of first order only. Therefore, their characteristics, which are

† For compressible materials, the velocity field equations assume a similar form—see Section 3 below.

again the slip lines, are related to the first spatial derivatives of the velocity, i.e. to the strain rates themselves.

On the other hand for a strain rate sensitive material, slip lines are no longer characteristics of the velocity field in the above sense, because the shear stress along a slip line is the shear yield limit k, which must, for reasons of equilibrium, be continuous across the slip line. However, in general it cannot be, because it depends on the strain rate, which may be different on either side of a characteristic. Exceptions would require a special form of the strain and strain rate sensitivity of the yield limit, showing, for instance, strain softening or rate softening. Let us consider the even more general case in which the characteristic represents also a surface of velocity discontinuity such that the corresponding equivalent strain-rate, i.e. $\overset{0}{\dot{\lambda}}$, is infinite. The equivalent strain $\overset{0}{\varepsilon}$ may be infinite or finite, depending on whether or not the discontinuity surface is a material surface. The condition for the possibility of the appearance of such a surface may then be written as

$$k(\overset{1}{\dot{\lambda}}, \overset{1}{\varepsilon}, \theta) = [k(\overset{0}{\dot{\lambda}}, \overset{0}{\varepsilon}, \theta) =]k(\overset{2}{\dot{\lambda}}, \overset{2}{\varepsilon}, \theta),$$

where θ is the (continuous) temperature, the superscripts 1 and 2 refer to the two sides of the surface, and the terms inside the brackets [. . .] may be omitted if there is no discontinuity of the velocity itself.

The above condition reflects the situation where both sides of the interface are in the plastic state. If, for instance, side No. 1 or 2 is rigid, then the conditions

$$k(0, \overset{1}{\varepsilon}, \theta) \geqslant [k(\overset{0}{\dot{\lambda}}, \overset{0}{\varepsilon}, \theta) =]k(\overset{2}{\dot{\lambda}}, \overset{2}{\varepsilon}, \theta) \quad \text{and} \quad k(\overset{1}{\dot{\lambda}}, \overset{1}{\varepsilon}, \theta)[= k(\overset{0}{\dot{\lambda}}, \overset{0}{\varepsilon}, \theta)] \leqslant k(0, \overset{2}{\varepsilon}, \theta)$$

would hold separately or jointly, respectively.

However, if at a surface of an incipient velocity discontinuity the above conditions cannot be fulfilled, the material must separate, thus allowing for different laws to hold at the interface (e.g. frictional laws, etc.). This sort of material separation has indeed been observed, e.g. at the boundary of a (rigid) dead metal zone in the metal extrusion process; see Becker *et al.* (1985, Fig. 7). Generally speaking, material interfaces of velocity discontinuity also represent interfaces of strain localization, which are themselves of interest.

Therefore, at the end of this paper the question of whether the characteristics of the velocity field equations might be considered as lines along which velocity discontinuities could be initiated will be discussed. This was indeed shown to be the case under plane stress conditions where, in uniaxial tension, the characteristics actually coincide with the localization bands (Lippmann, 1986). However, for plane strain or axial symmetry, it is doubtful that principal line characteristics have anything to do with velocity discontinuities.

Instead, for certain types of rate-sensitive material, another line or region in addition to the principal lines is found along which the equations degenerate in such a way that every direction becomes a characteristic direction. It could be that this kind of degeneration indeed leads to an incipient velocity jump.

2. BASIC NOTATION

The summation rule will be applied, the actual configuration envisaged, and the following basic quantities introduced: x^i, arbitrary curvilinear coordinates fixed in the space; $._{,i}$, $\partial/\partial x^i$; $|_i$, covariant derivative with respect to x^i; δ^k_i, Kronecker symbol; v^i, point velocities; $\lambda_i{}^k = \lambda^k{}_i = \frac{1}{2}(v_i|^k + v^k|_i)$, strain rates; $\sigma^i{}_k = \sigma_k{}^i$, Cauchy stresses; $\sigma = \sigma^i{}_i/3$, hydrostatic stress; $s^i{}_k = s_k{}^i = \sigma^i{}_k - \delta^i_k\sigma$, deviator of the Cauchy stress; f_i, static volume forces; t, time with t_0, initial value; θ, temperature; $Y = Y(\bar{\dot{\lambda}}, \bar{\varepsilon}, \theta)$, (uniaxial) yield limit, in which $\bar{\dot{\lambda}}$ is the equivalent strain rate, defined using $\Lambda = \sigma^i{}_k\lambda_i{}^k = Y\bar{\dot{\lambda}}$, the rate of work dissipated per unit volume, and $\bar{\varepsilon} = \bar{\varepsilon}_0 + \int_{t_0}^{t} \bar{\dot{\lambda}}\,dt$, the equivalent strain with $\bar{\varepsilon}_0$, the initial value.

The yield condition may be expressed in a standard manner as follows (Lippmann, 1981):

$$F(s^j{}_k) = Y,$$

F being a sufficiently smooth, symmetric and homogeneous function according to

$$F(s^j{}_k) = F(s_k{}^j)$$
$$F(\alpha s^j{}_k) = \alpha F(s^j{}_k); \quad \alpha \geqslant 0.$$

The associated yield rule then reads

$$\lambda_k{}^j = \bar{\lambda}\, \partial F/\partial s^k{}_j.$$

If the equivalent strain rate is expressed in terms of

$$\bar{\lambda} = G(\lambda_j{}^k) = G(\lambda^k{}_j), \tag{1}$$

the function G is easily seen to be sufficiently smooth, symmetric and homogeneous as well. Following Hill (1987), we have

$$s^j{}_k = F\, \partial G/\partial \lambda_j{}^k = Y\, \partial G/\partial \lambda_j{}^k = Y\, \partial G/\partial \lambda^k{}_j. \tag{2}$$

3. VELOCITY FIELD EQUATIONS OF RIGID-PLASTIC FLOW

Substituting eqn (2) into the static equilibrium equations with volume forces f_k acting, i.e.

$$s^j{}_k|_j + \sigma|_k + f_k = 0, \tag{3}$$

observing that

$$v_q|^p\, \partial G/\partial \lambda_q{}^p = v^q|_p\, \partial G/\partial \lambda^q{}_p = v^p|_q\, \partial G/\partial \lambda^p{}_q = v^p|_q\, \partial G/\partial \lambda_q{}^p$$

and introducing in addition to the hydrostatic stress σ the new variables

$$v^j{}_k = v^j|_k,$$

then using eqn (1), the following equations of plastic flow—to be called *velocity field equations*—are obtained:

$$\bar{L}^j{}_k{}^q{}_p v^p{}_q|_j + \frac{1}{Y}\sigma|_k = -N_k \tag{4}$$

$$N_k = \left(\frac{\partial Y/\partial \theta}{Y}\theta|_j + \frac{\partial Y/\partial \bar{\varepsilon}}{Y}\bar{\varepsilon}|_j\right)\frac{\partial G}{\partial \lambda_j{}^k} + f_k/Y$$

$$\bar{L}^j{}_k{}^q{}_p = h\frac{\partial G}{\partial \lambda_j{}^k}\frac{\partial G}{\partial \lambda_q{}^p} + \frac{\partial^2 G}{\partial \lambda_j{}^k\, \partial \lambda_q{}^p}; \quad h = \frac{\partial Y/\partial \bar{\lambda}}{Y}. \tag{5}$$

They must be complemented by the incompressibility condition, i.e.

$$v^j{}_j = \lambda_j{}^j = 0, \tag{6}$$

which allows us to eliminate one of the variables, and by a certain number of independent compatibility equations, i.e.

$$v^p{}_q|_r = v^p{}_r|_q, \tag{7}$$

such that there are altogether as many differential equations as there are variables, $v^p{}_q$ and σ.

For the Huber–von Mises flow law, the equivalent rate of strain may be expressed in terms of

$$G = [\tfrac{2}{3}\lambda_j{}^k\lambda_k{}^j]^{1/2} = [\tfrac{2}{3}\lambda^k{}_j\lambda^j{}_k]^{1/2}$$

and the coefficients from eqn (5) read as follows:

$$\bar{L}'_k{}^q{}_p = \frac{2}{3}\frac{1}{G}[\delta^q_k\delta^j_p + m\lambda_k{}^j\lambda_p{}^q]; \quad m = \tfrac{2}{3}(Gh-1)/G^2. \tag{8}$$

It should be added that for a compressible rigid–plastic material the velocity field equations assume a similar form, and can be derived as follows. In the yield condition, the yield rule and eqns (2) and (3), the stress deviator $s^k{}_j$ has to be replaced by the stress tensor $\sigma^k{}_j$ itself; while in eqns (3) and (4), the hydrostatic stress σ has to be removed as an independent variable.

4. CHARACTERISTICS

For plane plastic flow in Cartesian coordinates, $x^1 = x$ and $x^2 = y$ there are two independent compatibility conditions—eqn (7)—which can by virtue of eqn (6) be written as

$$v^1{}_1|_2 = v^1{}_2|_1; \quad v^2{}_1|_2 = -v^1{}_1|_1.$$

After eliminating $v^2{}_2$ with the aid of eqn (6) and lowering, in the given Cartesian coordinates, all indices, the above equations are rewritten, together with the two relevant velocity field equations (4), in matrix notation, as

$$\mathbf{R}\mathbf{w}_{,1} + \mathbf{S}\mathbf{w}_{,2} = -\mathbf{N} \tag{9}$$

$$\mathbf{w} = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{21} \\ \sigma \end{bmatrix}, \quad \mathbf{N} = \tfrac{2}{3}G\begin{bmatrix} 0 \\ 0 \\ N_1 \\ N_2 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1+m(\lambda_{11}-\lambda_{22})\lambda_{11} & m\lambda_{12}\lambda_{11} & m\lambda_{21}\lambda_{11} & 3G/2Y \\ m(\lambda_{11}-\lambda_{22})\lambda_{21} & 1+m\lambda_{12}\lambda_{21} & m\lambda_{21}\lambda_{21} & 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m(\lambda_{11}-\lambda_{22})\lambda_{12} & m\lambda_{12}\lambda_{12} & 1+m\lambda_{21}\lambda_{12} & 0 \\ -1+m(\lambda_{11}-\lambda_{22})\lambda_{22} & m\lambda_{12}\lambda_{22} & m\lambda_{21}\lambda_{22} & 3G/2Y \end{bmatrix}.$$

In order to find the angle α between a characteristic and, for instance, the x^2 direction, the characteristic equation is written (cf. Lippmann, 1981) as

$$D(\alpha) = \det(\mathbf{R}\cos\alpha - \mathbf{S}\sin\alpha) = 0,$$

i.e., using $\lambda_{12} = \lambda_{21}$,

$$D = -m\frac{3G}{2Y}[\lambda_{12}\cos(2\alpha) + \tfrac{1}{2}(\lambda_{11} - \lambda_{22})\sin(2\alpha)]^2.$$

The solutions are:

$$\tan(2\alpha) = -2\lambda_{12}/(\lambda_{11} - \lambda_{22}) \quad \text{(twofold)}, \tag{10a}$$

$$\alpha \text{ arbitrary if } m = 0, \quad \text{i.e. } hG = 1. \tag{10b}$$

The principal lines therefore represent a pair of families of characteristics, each counting twofold. In addition, if condition (10b) is fulfilled, then, because of eqn (5) and as $G = \bar{\lambda}$, the following relation holds:

$$\partial(Y/\bar{\lambda})/\partial\bar{\lambda} = 0. \tag{11}$$

Equation (11) defines a point, a line or a region of degeneration of the velocity field equations where each direction is a characteristic direction. Of course, this holds true also for the direction of a degeneration line itself.

In the event of an axially symmetric rigid-plastic deformation, the coordinates may be identified with the axial coordinate, $x^1 = z$, the radius, $x^2 = r$, or the azimuthal angle, $x^3 = \psi$, respectively. Since $v^3 \equiv 0$ and $v^1{}_{,3} \equiv v^2{}_{,3} \equiv 0$, the velocity field equations depend on v^1 and v^2 only, whereas the covariant derivatives

$$v^j|_k = v^j{}_{,k} + \Gamma^j_{ik}v^i,$$

where Γ^j_{ik} are the Christoffel symbols, differ from the partial derivatives only by undifferentiated terms. Therefore, if in eqns (4), (6) and (7) the covariant derivatives are replaced by the partial ones, and if now besides σ the variables $v^k{}_j = v^k{}_{,j}$ are introduced, the matrix equations (10) remain valid with a modified right-hand side. This does not affect the calculation of the characteristic directions, so eqns (10) and (11) remain valid also.

5. CONCLUSIONS

The velocity field equations have been formulated for a standard rigid-plastic incompressible material. They hold in a similar form also for compressible materials, and are valid not only for ideally plastic or strain hardening plastic solids, but also for strain rate and temperature sensitive bodies.

The velocity field equations allow, together with the equations of heat generation and heat convection/conduction (cf. Becker *et al.*, 1985, 1987), to determine the velocity field and the hydrostatic stress directly, without considering the stress components. The characteristics presented under conditions of plane strain or axial symmetry therefore have nothing to do with the corresponding classical slip line or principal line characteristics.

In all cases there exist two families of characteristics, each counting twofold—i.e. as many families as there are kinematic and static unknowns in the problem. Although the so-called characteristic equations, valid along the characteristics themselves, have not been determined, one may assume that the mechanical part of the problem (without the thermal part) is hyperbolic. As this holds already at the first instant of the deformation, the characteristics do not supply any criterion for the onset of strain localization which, by general experience, does not start at the very beginning. Also, because the characteristics were found to coincide with the principal lines, i.e. with the trajectories of principal strain-rate, they can hardly correspond to jump lines for the velocity or for the displacement, which should both have a jump component parallel to the line.

In addition, another line, called the degeneration line, may exist, along which every direction is a characteristic direction. This line may also be a point or an entire region. It does not form before the strain rate reaches a certain threshold value, given by eqn (11). This threshold may indeed indicate the onset of localization, although it is not clear whether it is sufficiently realistic for technological materials. Let us illustrate the situation by comparing two very similar (temperature-independent) hardening laws often used in metal plasticity:

$$Y = Y_0[1+\varepsilon^{-\beta}(T\bar{\lambda})^{\mu}] \quad \text{or} \quad Y = Y_0\varepsilon^{-\beta}(T\bar{\lambda})^{\mu},$$

where Y_0, T, β and μ are positive constants. Both laws differ for small strains or strain rates only, i.e. in a regime where the experimental determination of the parameters is difficult anyway. Nevertheless, they show basically different behaviour with respect to the degeneration criterion (11) which now becomes

$$(\mu-1)\varepsilon^{-\beta}(T\bar{\lambda})^{\mu} = 1 \quad \text{or} \quad (\mu-1)\varepsilon^{-\beta}(T\bar{\lambda})^{\mu-2} = 0.$$

The first relation can easily be fulfilled if $\mu > 1$, while the second is valid only if $\mu = 1$. However, in that case it holds identically, such that it loses its significance as a criterion (this holds true in particular for a Newtonian fluid).

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